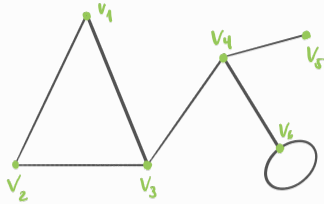


Lecture 1

§0. Basic concepts.

Definition: A **graph** $G=(V,E)$ is a set of vertices $V=\{v_1, \dots, v_n\}$ and edges that are sets of 2 vertices $\{u,v\}$ that are connected.

example:



$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

$$E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_6, v_6\}\}$$

We say a graph is **simple** if it has no parallel edges (a parallel edge is ) and no loops (a loop is an edge $\{v,v\}$ like $\{v_6, v_6\}$ in the previous example)

⚠ Unless stated otherwise, we assume all graphs to be simple.

A **path** in a graph is a sequence $(v_0, e_1, v_1, \dots, e_n, v_n)$, $v_i \in V$, $e_i \in E \quad \forall i$ such that $e_i = \{v_{i-1}, v_i\}$. If $v_0 = v_n$, then we call it a **cycle**.

A graph is **connected** if $\forall v, w \in V$ there is a path in G with $v_0 = v$ and $v_n = w$.

Next, we will learn some tools that we will use going forward.

§0.1 Proof by induction

To prove something by induction there are 2 steps:

- 1) Base case: Prove the claim holds for the smallest case (usually $n=1$)
- 2) Induction step: We assume the claim is true for n (induction hypothesis) and prove it for $n+1$

Example: Prove that $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Proof:

Base case: For $n=1$ clearly $1^2 = \frac{1 \cdot (1+1) \cdot (2 \cdot 1 + 1)}{6} = \frac{6}{6} = 1$ ☺

Inductive step: The induction hypothesis (IH) is that $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$, we want to prove that $1^2 + \dots + n^2 + (n+1)^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$

$$\begin{aligned} \underbrace{1^2 + 2^2 + \dots + n^2}_{\text{we can apply IH}} + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\ &= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

this is what we wanted to prove ☺

§ 0.2 Pidgeonhole principle

Theorem (Pidgeonhole principle): Let $n, k \in \mathbb{N}$ so that $n > k$. If we have n identical objects we want to put in k boxes, then there will be at least one box with 2 objects.

Example: Prove that there is a number divisible by 2025 in the sequence $7, 77, 777, 7777, \dots$

Proof

Lets consider the first 2025 numbers in the sequence and we create a list of the remainders when we divide them by 2025.

Then we have 2 cases

- 1) There is a 0 in the list $\Rightarrow$ one of the first 2025 Terms of the sequence is divisible by 2025
- 2) There are no 0's, so there are 2024 possible remainders. By pidgeonhole principle, there must 2 terms with the same remainder, suppose they have length $i, j \in \mathbb{N}$ with $j > i$, so $\underbrace{777\dots 77}_{i \text{ 7's}}$ and $\underbrace{777\dots 77}_{j \text{ 7's}}$

Then $\underbrace{77\dots 77}_i - \underbrace{77\dots 7}_i = \underbrace{77\dots 70\dots 0}_{j-i}$ is divisible by 2025
and $7\dots 70\dots 0 = 7\dots 7 \cdot 10^i \rightarrow$ this is not divisible by 2025 so $\underbrace{7\dots 7}_{j-i}$ must be divisible by 2025 and it is part of the sequence.