

## Lecture 2 - degree sequence

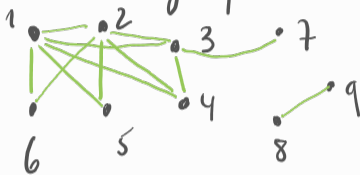
Definition: Let  $G=(V, E)$  be a graph and  $v \in V$ , its **degree** is the amount of edges in  $E$  containing  $v$ . The **degree sequence** of  $G$  is the tuple consisting of the degrees of all vertices in decreasing order.

**example**:  has degree sequence  $(3, 2, 2, 1)$ .

**Question**: Construct graphs with the following degree sequences:

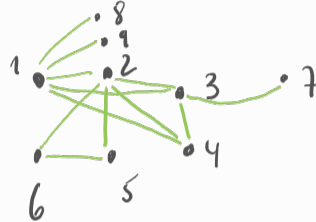
$(5, 5, 4, 3, 2, 2, 1, 1, 1)$  ✓,  $(8, 2, 2, 2, 1, 1, 1, 1, 1)$  ✗,  $(6, 5, 4, 4, 2, 1)$ ,  $(3, 3, 1, 1)$

is it always possible?



handshake lemma

or



first vertex is impossible

✗ Erdős Gallai

or ...

When is a degree sequence feasible? What does the degree sequence tell us about the graph?

Handshake lemma: The number of vertices with odd degree is even.

Pf

$\sum_{v \in V} \deg(v) = 2|E|$ , i.e. the sum is even. Let  $V_{\text{even}}$  be vertices with even degree  $V_{\text{odd}}$  the vertices with odd degree

$$\Rightarrow \sum_{v \in V} \deg(v) = \sum_{v \in V_{\text{even}}}^{\text{even}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v)$$

If  $\deg(v_i) = 2n_i + 1 \quad \forall v_i \in V_{\text{odd}}$

$$\Rightarrow \sum_{v \in V_{\text{odd}}} \deg(v_i) = 2 \sum_{v \in V_{\text{odd}}} n_i + |V_{\text{odd}}|$$

$\therefore$  For the sum to be even  $|V_{\text{odd}}|$  has to be even



Definition: We say a graph is **eulerian** if it has a cycle that includes all the edges exactly once, and through every vertex at least once

Theorem A graph is eulerian iff it is connected and all degrees are even

Pf *not necessarily simple*

$\Rightarrow$  easy, count degree as we go through the cycle

$\Leftarrow$  Induction on # of edges

What happens if we just want an eulerian path? (not closed)

A cycle is *Hamiltonian* if it goes through every vertex exactly once

There is no "nice" characterization of graphs with a Hamiltonian cycle.

Determining this is NP-complete.

Theorem Let  $G$  be simple on  $n \geq 3$  vertices such that all of them have degree at least  $n/2$ . Then  $G$  has a Hamiltonian cycle

Proof

Imagine there are graphs such that all vertices have degree at least  $n/2$  and there is no Hamiltonian cycle. Then we can add all possible edges such that it is still not Hamiltonian, we call the graph with the added edges  $G$ . Since we added all possible edges, we know that adding any extra edge now would create a Hamiltonian cycle, let's call this extra edge  $\{y, z\}$  and the Hamiltonian cycle it creates

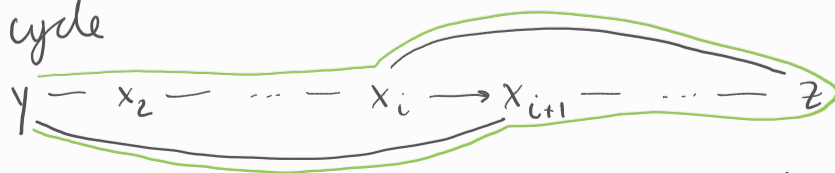
$y - x_2 - x_3 - \dots - x_{n-1} - z$

Consider the sets

$S = \{i : 1 \leq i \leq n-1 \text{ and } \{y, x_{i+1}\} \text{ is an edge}\}$

$T = \{i : 1 \leq i \leq n-1 \text{ and } \{x_i, z\} \text{ is an edge}\}$

Both of them are subsets of  $\{1, \dots, n-1\}$  and  $|S| = \deg(y) \geq n/2$  and  $|T| = \deg(z) \geq n/2$  so they must intersect. Let  $i \in S \cap T$ , then we can construct the cycle



which is a Hamiltonian cycle in  $G$   $\rightarrow \leftarrow$  contradiction!

Finally, we will see a theorem that characterizes when a degree sequence is possible.

Theorem (Erdős - Gallai): Let  $(d_1, \dots, d_n)$  be a decreasing sequence in  $\mathbb{N}$ . It is the degree sequence of a simple graph if and only if the sum of the degrees is even and for every  $k \in \{1, 2, \dots, n\}$  we have

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(k, d_i)$$

Proof

1) Let's first prove that every simple graph satisfies the inequality. Let  $(d_1, \dots, d_n)$  be the degree sequence of a simple graph.

By the handshake lemma, we know the sum of the degrees is even and let  $k \in \{1, \dots, n\}$ , we call  $G_k$  the subgraph of  $G$  consisting of only the first  $k$  vertices and edges in  $G$  between them, then

$$\sum_{i=1}^k d_i = \sum_{v \in G_k} \deg(v) + \text{number of edges going out of } G_k$$

notice that the maximum number of edges in  $G_k$  would be if  $G_k$  was complete, so  $\sum_{v \in G_k} \deg(v) \leq k(k-1)$   
# of vertices      max possible degree of each vertex

and for every vertex outside of  $G_k$ , there can be no more than  $k$  edges going to  $G_k$  and also no more edges than its degree, so

$$\text{number of edges going out of } G_k \leq \sum_{i=k+1}^n \min\{k, d_i\}.$$

Putting both inequalities together we get

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(k, d_i)$$

2) Now let's prove that there is a sequence  $(d_1, \dots, d_n)$  that satisfies

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(k, d_i)$$

and the sum of the  $d_i$ 's is even. We prove that there exists a simple

graph with that degree sequence by induction on the sum of the degrees

→ Base case: Suppose  $\sum_{i=1}^n d_i = k$ , then the sequence could only be (1,1) or (2)  
notice that the sequence (2) does not satisfy

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(k, d_i) \text{ when } k=1 \text{ because}$$

$$2 \not\leq 1 \cdot (1-1)^0$$

so the only valid sequence is (1,1) and there is a simple graph with that degree sequence:  $\bullet - \bullet$

→ Induction step: Let  $(d_1, \dots, d_n)$  be a sequence such that it satisfies

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(k, d_i)$$

and  $\sum_{i=1}^k d_i = s+2$  is even

**Induction hypothesis:** If  $(b_1, \dots, b_n)$  is such that  $\sum_{i=1}^k b_i = s$  and

$$\sum_{i=1}^k b_i \leq k(k-1) + \sum_{i=k+1}^n \min(k, b_i) \text{ for every } k$$

then there is a simple graph with that degree sequence

The idea now is to lower the sequence  $(d_1, \dots, d_n)$  so that the sum is  $s$  and it still satisfies the inequality for every  $k$ . To do this, let  $t$  be the smallest number such that  $d_{t+1} < d_t$  (the first "step down") then we consider the new sequence

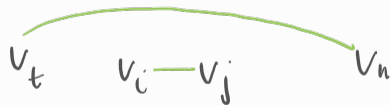
$$(d_1, d_2, \dots, d_{t-1}, \underline{d_t-1}, d_{t+1}, \dots, d_{n-1}, \underline{d_n-1})$$

the sum of this sequence is  $(s+2) - 2 = s$  and it still satisfies the inequality for every  $k$  (exercise: move the inequality still holds)  
then by our induction hypothesis, there is a simple graph  $G$  with that degree sequence.

To get a graph with the original degree sequence we have 2 cases

1) there is no edge in  $G$  between the vertices with degrees  $d_t-1$  and  $d_n-1$ : then we can add an edge between the two of them and we get the graph we wanted

2) There is an edge in  $G$  between the vertex with degree  $d_t-1$  and  $d_n-1$ : We call the vertices  $v_t$  and  $v_n$  respectively, then we find  $v_i, v_j$  such that



and we change it to  $v_t$  —  $v_i$        $v_j$  —  $v_n$

notice this doesn't change the degrees, and now we are back in case 1. ■