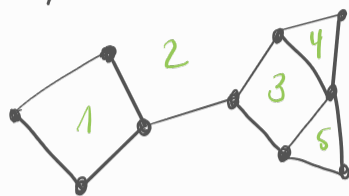


* notation
we will use V and E
to mean number of vertices
and edges

Lecture 3: Planar graphs

Definition: Let G be a graph that can be drawn on a plane surface so that no two edges intersect. Then G is a **planar graph**.

If a graph is planar, then it divides the plane into regions, these are called **faces**.



← This graph has 5 faces

Question for today: How do we know if graph is planar?

Exercise: Try to see what complete graphs are planar, how many edges do I need to take away to make them planar?

The following theorem will help us detect planar graphs:

Theorem [Euler]: Let G be a connected planar graph with V vertices, E edges and F faces. Then $V + F = E + 2$.

Before proving it let's see a particular example: **trees**

Definition: a **tree** is a connected graph with no cycles.

Lemma: A tree has at least two vertices with degree 1 (if $|V| \geq 2$)

Pf

Pick a vertex v of the tree, since there are more vertices and trees are connected, then we can start a path on v by picking a random neighbor. We continue this way until we can't add a new vertex, since there are no cycles, this means the final vertex had degree 1.



So we found 1 degree 1 vertex. To find the second we have two options

- 1) v had degree 1, so v and w are the two vertices we wanted
- 2) v has degree ≥ 2 , so we can start a new path with another neighbor

in the same way as before and find another degree 1 vertex. 

Now we can prove that the theorem holds for trees

Proposition: A tree with n vertices has $n-1$ edges.

Pf

Induction on $|V|$

Base case: If there is only one vertex the tree is $\bullet$ which has 0 edges.

Induction step: We want to prove the proposition for n vertices

Induction hypothesis: A tree with $n-1$ vertices has $n-2$ edges

By the previous lemma, we know that in a tree G with $n \geq 2$ vertices there is a vertex with degree 1. If we remove that vertex with its edge, we get a new tree with $n-1$ vertices and one edge less than G .

By IH, the new tree has $n-2$ edges, so G has $n-1$ edges. 

This proves the theorem for trees because trees are connected and planar (exercise: why are they planar? prove by induction) and they have only 1 face so $V+F = n+1 = (n-1)+2 = E+2$.

Now we can prove Euler's theorem! 

Historic note: Francesco Maurolico stated the theorem for graphs of platonic solids in 1537 and Euler stated it for polytopes in 1758, but never proved it

Proof (Euler's theorem):

Induction on number of edges

Base case: If there is one edge, then the graph is $\bullet \text{---} \bullet$ or $\odot$

In the first one $2+1 = 1+2$  and in the second $1+2 = 1+2$ 

Induction step: Let G be a planar, connected graph with E edges

Induction hypothesis: A connected planar graph with E' edges with $E' = E - 1$, V' vertices and F' faces satisfies $V' + F' = E' + 2$

We have 2 cases:

1) G has an edge e such that it is part of a cycle:

In this case, taking out e of G gives us a new connected planar graph G' with $V' = V$, $E' = E - 1$ and $F' = F - 1$. By IH,

$$V' + F' = E' + 2 \Rightarrow V + F - 1 = E - 1 + 2 \quad / + 1$$

$$\Rightarrow V + F = E + 2 \quad \checkmark$$

2) If G has no edge that is in a cycle, then it is a tree and we already proved the theorem holds $\checkmark$

Now we can prove some graphs are not planar.

Example 1: $K_{3,3}$ is not planar.

We know that $K_{3,3}$ has 6 vertices and $\frac{3 \cdot 6}{2} = 9$ edges, so if it was planar, it would have $F = 9 + 2 - 6 = 5$ faces.

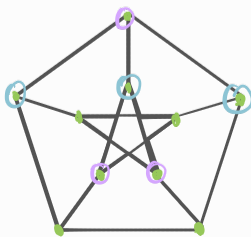
Notice each edge is contained in 2 faces and the faces should be quadrilateral, so 5 faces implies at least $\frac{4 \cdot 5}{2} = 10$ edges $\times$

Example 2: K_5 is not planar

Again we know $V = 5$, $E = \frac{4 \cdot 5}{2} = 10$, so if K_5 was planar it would have $F = 10 + 2 - 5 = 7$ faces. Each of them has at least 3 edges, so the graph would need at least $\frac{7 \cdot 3}{2} = 10.5$ edges $\times$.

Kuratowski's theorem: A graph is not planar if and only if it contains a subgraph that is a subdivision of K_5 or $K_{3,3}$

Example:



contains a subdivision
of $K_{3,3}$

